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Low Reynolds numbers flow past an ellipsoid of revolution of large aspect ratio

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The results of Proudman & Pearson (1957) and Kaplun & Lagerstrom (1957) for a sphere and a cylinder are generalized to study an ellipsoid of revolution of large aspect ratio with its axis of revolution perpendicular to the uniform flow at infinity. The limiting case, where the Reynolds number based on the minor axis of the ellipsoid is small while the other Reynolds number based on the major axis is fixed, is studied. The following points are deduced: (1) although the body is three-dimensional the expansion is in inverse power of the logarithm of the Reynolds number as the case of a two-dimensional circular cylinder; (2) the existence of the ends and the variation of the diameter along the axis of revolution have no effect on the drag to the first order; (3) a formula for drag is obtained to higher order.

1. Introduction

Viscous flow at low Reynolds numbers past two-dimensional or three-dimensional objects has been studied extensively (cf. Oseen 1927 and Lamb 1932) with the aid of Stokes's and Oseen's equations. Recent work by Proudman & Pearson (1957) and Kaplun & Lagerstrom (1957) has clarified the relation of these solutions to asymptotic expansions of the Navier–Stokes equations. These solutions exhibit a marked difference between the two-dimensional and three-dimensional case. The results presented in this paper are based on one of the examples discussed by the author (Shi 1963) for the purpose of clarifying this difference, in particular by studying in detail the transition from the three-dimensional case to the two-dimensional case. For this purpose, flow-past bodies of large aspect ratio, i.e. bodies whose extension transverse to the flow is much larger than that parallel to the flow, is considered. In this paper we study the case of uniform flow past an ellipsoid of revolution whose half-axis parallel to the uniform flow at infinity is denoted by λ and whose half-axis perpendicular to the uniform flow at infinity is denoted by L. Two Reynolds numbers may be formed, namely, $\mathrm{Re} = U\lambda/\nu$ and $Re = UL/\nu$. We study the limiting case of Re tending to zero, Re being fixed. This clearly indicates the body is of large aspect ratio. In a recent paper, Breach (1961) has also studied all ellipsoids of revolution both prolate and oblate. His solutions, however, are valid only when the Rey-

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nolds number based on the major axis tends to zero, while in our case the Reynolds number based on the major axis is not necessarily small. Furthermore, he only considers spheroids whose axis of symmetry is parallel to the undisturbed stream at infinity. In our case, the axis of revolution of the ellipsoid is perpendicular to the undisturbed stream at infinity. The method used here essentially follows the spirit of Proudman & Pearson (1957) and Kaplun & Lagerstrom (1957). The detailed expansion procedures and matching principles follow the work of Lagerstrom & Cole (1955) and Kaplun & Lagerstrom (1957). The present problem does not have rotational symmetry about an axis parallel to the uniform flow at infinity and, therefore, there exists no Stokes's stream function. We obtain directly the expansions for velocity and pressure. The present problem can be solved by introducing several (more than two) simultaneous expansions, i.e. an 'outer expansion', a 'shank expansion' and two 'end expansions'. The proper choice of variables for each expansion was discussed in detail by Shi (1963). The details are not repeated here. The details of the expansion procedures and the matching between them present certain interest, although certainly no new principles need to be introduced. In the present case, the expansion procedures can be exhibited in detail and higher-order terms can be obtained. It is also of interest that the idea of an intermediate (rather than inner) expansion proposed by Lagerstrom & Kaplun (1957) is intimately involved and quite helpful in the present case.

2. Formulation

We consider viscous incompressible flow past an ellipsoid of revolution of large aspect ratio. The governing equations are the Navier–Stokes equations which in dimensional form are

$$(\mathbf{q}.\nabla)\,\mathbf{q}+\rho^{-1}\nabla p = \nu\nabla^2\mathbf{q}, \tag{1a}$$

$$\nabla.\mathbf{q} = 0, \tag{1b}$$

with the boundary conditions

$$\mathbf{q} = 0 \quad \text{on the body,} \tag{1c}$$

$$\mathbf{q} = U\mathbf{i}; \quad p = p_\infty \quad \text{at infinity,} \tag{1d}$$

where $\mathbf{i}$ is the unit vector in the x-direction and the body is defined by

$$r^2/\lambda^2+z^2/L^2 = 1 \tag{2a}$$

and $r^2 = x^2+y^2$. The other symbols in equations (1 a–d) have their usual meanings in fluid mechanics. If Oseen or outer variables (i.e. $\tilde{x}_i = Ux_i/\nu$) are used the body becomes

$$\tilde{r}^2/\mathrm{Re}^2+\tilde{z}^2/Re^2 = 1. \tag{2b}$$

Then for low Reynolds number flow (in the limit, $\mathrm{Re}\to 0$) the body shrinks to a needle of zero radius. Thus the outer limit or the leading term of the outer expansion (cf. equation (3d)) is $\mathbf{g}_0 = \mathbf{i}$. Therefore, the outer flow (the flow far from the body) is governed by the Oseen equations (cf. equations (4a, b)) as expected. But this outer limit does not satisfy the boundary condition near the body (i.e. $\mathbf{q} = 0$). Different limits must be introduced near the body as in the case of a sphere or cylinder. In the present case, three different inner limits are needed because the inner limit is not uniform near the body. A shank limit is introduced to study the flow near the body but not near the ends. A left-end limit and a right-end limit are introduced to study the flow near the left and right ends. respectively.

3. Limits, expansions and associated equations

In this section, we shall define the various limits needed in various regions. Proper expansions will then be introduced and the associated equations will be established in the corresponding regions.

(*a*) *Outer*. As independent variables (outer variables) we use

$$\tilde{x}_i = Ux_i/\nu. \tag{3a}$$

The dependent variables are $\mathbf{q}^* = \mathbf{q}/U$ and $p^* = (p-p_\infty)/\rho U^2$. The outer limit is defined as the limit $\mathrm{Re}\to 0$ for Re, $\tilde{x}_i$ fixed. Thus the Navier–Stokes equations can be written in terms of outer variables as

$$(\mathbf{q}^*.\tilde{\nabla})\mathbf{q}^* + \tilde{\nabla}p^* = \tilde{\nabla}^2\mathbf{q}^*, \tag{3b}$$

$$\tilde{\nabla}.\mathbf{q}^* = 0, \tag{3c}$$

where $\tilde{\nabla}$ and $\tilde{\nabla}^2$ are in terms of outer variables. The outer expansions for velocity and pressure are assumed to have the form

$$\mathbf{q}^* = \mathbf{g}_0 + \epsilon\mathbf{g}_1 + \epsilon^2\mathbf{g}_2 + \ldots, \tag{3d}$$

$$p^* = \epsilon p_1^* + \epsilon^2 p_2^* + \ldots, \tag{3e}$$

where ϵ is to be defined later.

The outer limit is $\mathbf{g}_0 = \mathbf{i}$ in the present study. Therefore the governing equations for $\mathbf{g}_1$ are the Oseen equations

$$\partial\mathbf{g}_1/\partial\tilde{x} + \tilde{\nabla}p_1^* = \tilde{\nabla}^2\mathbf{g}_1, \tag{4a}$$

$$\tilde{\nabla}.\mathbf{g}_1 = 0 \tag{4b}$$

and the governing equations for $\mathbf{g}_n (n \geqslant 2)$ are

$$\frac{\partial\mathbf{g}_n}{\partial\tilde{x}} + \tilde{\nabla}p_n^* = \nabla^2\mathbf{g}_n - \sum_{i=1}^{n-1}(\mathbf{g}_i.\tilde{\nabla})\,\mathbf{g}_{n-i}, \tag{5a}$$

$$\tilde{\nabla}.\mathbf{g}_n = 0. \tag{5b}$$

(*b*) *Shank*. In the shank region, $-Re < \tilde{z} < Re$, we use the following independent variables

$$x^* = x/\lambda = \tilde{x}/\mathrm{Re}, \quad y^* = \tilde{y}/\mathrm{Re}, \quad \tilde{z} = Uz/\nu. \tag{6a}$$

The shank limit is defined as the limit $\mathrm{Re}\to 0$ for x^*, y^*, $\tilde{z}$ and Re fixed. The dependent variables are

$$\mathbf{q}^* = \mathbf{q}/U = \mathbf{i}U^* + \mathbf{j}V^* + \mathbf{k}W^*, \tag{6b}$$

$$p^+ = \lambda(p-p_\infty)/\rho U. \tag{6c}$$

The Navier–Stokes equations in terms of shank variables are as follows:

$$\mathrm{Re}\, U^*\frac{\partial U^*}{\partial x^*}+\mathrm{Re}\, V^*\frac{\partial U^*}{\partial y^*}+\mathrm{Re}^2\, W^*\frac{\partial U^*}{\partial \tilde{z}}+\frac{\partial p^+}{\partial x^*} = \frac{\partial^2 U^*}{\partial x^{*2}}+\frac{\partial^2 U^*}{\partial y^{*2}}+\mathrm{Re}^2\frac{\partial^2 U^*}{\partial \tilde{z}^2}, \tag{7a}$$

$$\mathrm{Re}\, U^*\frac{\partial V}{\partial x^*}+\mathrm{Re}\, V^*\frac{\partial V^*}{\partial y^*}+\mathrm{Re}^2\, W^*\frac{\partial V^*}{\partial \tilde{z}}+\frac{\partial p^+}{\partial y^*} = \frac{\partial^2 V^*}{\partial x^{*2}}+\frac{\partial^2 V^*}{\partial y^{*2}}+\mathrm{Re}^2\frac{\partial^2 V^*}{\partial \tilde{z}^2}, \tag{7b}$$

$$\mathrm{Re}\, U^*\frac{\partial W^*}{\partial x^*}+\mathrm{Re}\, V^*\frac{\partial W^*}{\partial y^*}+\mathrm{Re}^2\, W^*\frac{\partial W^*}{\partial \tilde{z}}+\mathrm{Re}\frac{\partial p^+}{\partial \tilde{z}} = \frac{\partial^2 W^*}{\partial x^{*2}}+\frac{\partial^2 W^*}{\partial y^{*2}}+\mathrm{Re}^2\frac{\partial^2 W^*}{\partial z^{*2}}, \tag{7c}$$

$$\frac{\partial U^*}{\partial x^*}+\frac{\partial V^*}{\partial y^*}+\mathrm{Re}\frac{\partial W^*}{\partial \tilde{z}} = 0. \tag{7d}$$

The limiting equations for $\mathrm{Re} \rightarrow 0$ are

$$\frac{\partial^2 U^*}{\partial x^{*2}}+\frac{\partial^2 U^*}{\partial y^{*2}} = \frac{\partial p^+}{\partial x^*}, \tag{8a}$$

$$\frac{\partial^2 V^*}{\partial x^{*2}}+\frac{\partial^2 V^*}{\partial y^{*2}} = \frac{\partial p^+}{\partial y^*}, \tag{8b}$$

$$\frac{\partial^2 W^*}{\partial x^{*2}}+\frac{\partial^2 W^*}{\partial y^{*2}} = 0, \tag{8c}$$

and

$$\frac{\partial U^*}{\partial x^*}+\frac{\partial V^*}{\partial y^*} = 0. \tag{8d}$$

The intermediate shank expansions of velocity and pressure are assumed to have the form

$$\mathbf{q}^* = \mathbf{u}_0+\epsilon u_1+\epsilon^2 u_2+\ldots, \tag{9a}$$

$$p^+ = p_0+\epsilon p_1+\epsilon^2 p_2+\ldots, \tag{9b}$$

where $\mathbf{u}_0$, p_0, etc., are intermediate shank solutions which are functions of x^*, y^*, $\tilde{z}$, Re and Re. The governing equation for $\mathbf{u}_0$ and p_0, $\mathbf{u}_1$ and p_1, etc., are equations (8*a–d*).

In terms of shank variables, the body can be expressed as

$$r^{*2}+\tilde{z}^2/Re^2 = 1. \tag{10}$$

At the shank limit, the body is then a finite ellipsoid of revolution.

(*c*) *Left end.* By symmetry, we shall concentrate on discussing the left end. The left-end variables are defined as

$$x^+ = \tilde{x}/\mathrm{Re}^2, \quad y^+ = \tilde{y}/\mathrm{Re}^2, \quad z^+ = (\tilde{z}+Re)/\mathrm{Re}^2 = \bar{z}/\mathrm{Re}^2. \tag{11}$$

The left-end limit is defined as the limit $\mathrm{Re} \rightarrow 0$ for Re, x^+, y^+ and z^+ fixed. In terms of left-end variables, the body is of the form

$$r^{+2} = \frac{2z^+}{Re}-\frac{\mathrm{Re}^2 z^{+2}}{Re^2} = \frac{2z^+}{Re}+O(\mathrm{Re})^2. \tag{12}$$

Thus at the left-end limit, the body becomes a semi-infinite paraboloid. If we define

$$\tau = \tfrac{1}{2}\{[r^{+2} + (z^{+} - \tau_0)^2]^{\frac{1}{2}} - (z^{+} - \tau_0)\}, \tag{13a}$$

then, to the order of Re^2, the body is

$$\tau = \tau_0 = 1/2Re. \tag{13b}$$

The intermediate left-end expansion is

$$\mathbf{q}^* = \mathbf{v}_0 + \epsilon \mathbf{v}_1 + \epsilon^2 \mathbf{v}_2 + \dots, \tag{14a}$$

$$p^{++} = p_0' + \epsilon p_1' + \epsilon^2 p_2' + \dots. \tag{14b}$$

The governing equations for $\mathbf{v}_n$ $(n \geqslant 0)$ are three-dimensional Stokes equations and $p^{++} = \mathrm{Re}\, p^{+} = \mathrm{Re}^2 p^*$.

(*d*) *Right end.* By symmetry, the right-end variables are defined as

$$x^{+} = \tilde{x}/\mathrm{Re}^2, \quad y^{+} = \tilde{y}/\mathrm{Re}^2, \quad \bar{z}^{+} = (\tilde{z} - Re/\mathrm{Re}^2). \tag{15}$$

Then the body may be expressed in terms of right-end variables as

$$\tau^{+} = \tau_0 = 1/2Re, \tag{16a}$$

where

$$\tau^{+} = \tfrac{1}{2}\{(\bar{z}^{+} + \tau_0) + [(\bar{z}^{+} + \tau_0)^2 + r^{+2}]^{\frac{1}{2}}\}. \tag{16b}$$

The intermediate right-end expansion is

$$\mathbf{q}^* = \mathbf{w}_0 + \epsilon \mathbf{w}_1 + \epsilon^2 \mathbf{w}_2 + \dots, \tag{17a}$$

$$p^{++} = p_0'' + \epsilon p_1'' + \epsilon^2 p_2'' + \dots. \tag{17b}$$

In all the above expansions, ϵ is defined as

$$\epsilon(\mathrm{Re}) = \frac{1}{\log(4/\mathrm{Re}) - \gamma + \frac{1}{2}}, \tag{17c}$$

the significance of which will be apparent later.

4. Determination of solutions

Determination of $\mathbf{g}_0$. As discussed previously, we know that the principal limit is

$$\mathbf{g}_0 = \mathbf{i}. \tag{18}$$

Determination of $\mathbf{u}_0$. The intermediate shank solution is determined as

$$\begin{aligned} \mathbf{u}_0 &= \epsilon(\mathrm{Re})\, \mathbf{h}_1 \\ &= \epsilon(\mathrm{Re}) \left\{ \mathbf{i} \left[\log r^* - \tfrac{1}{2} \log \left(\frac{Re^2 - \tilde{z}^2}{Re^2} \right) + \frac{1}{2} \right] \right. \\ &\qquad \left. - x^* \nabla^* \log r^* - \frac{1}{2} \frac{Re^2 - \tilde{z}^2}{Re^2} \nabla^* \left(\frac{x^*}{r^{*2}} \right) \right\}. \end{aligned} \tag{19a}$$

The corresponding pressure term p_0 is obtained as

$$p_0 = -\epsilon(\mathrm{Re})\, 2x^*/r^{*2}. \tag{19b}$$

It is obvious that $\mathbf{u}_0$ satisfies the governing equations (8) and the boundary condition

$$\mathbf{h}_1 = 0 \text{ on the body} \quad (\text{i.e. } r^{*2} = (Re^2 - \tilde{z}^2)/Re^2). \tag{19c}$$

Then, by the matching condition

$$\lim_{\substack{f \\ \mathrm{Re}\to 0}} |\mathbf{i}-\mathbf{u}_0| = 0, \tag{20}$$

for ord f in some overlap domain, we can determine $\epsilon(\mathrm{Re})$. To apply the matching condition (20) we write $\mathbf{u}_0$ in terms of x_f, y_f and $\tilde{z}$, where x_f and y_f are defined as

$$x_f = \tilde{x}/f(\mathrm{Re}), \quad y_f = \tilde{y}/f(\mathrm{Re}); \tag{21a}$$

then
$$\mathbf{u}_0 - \mathbf{i} = \mathbf{i}\left(\epsilon \log\frac{1}{\mathrm{Re}} - 1\right) + \epsilon\left[\mathbf{i}\left\{\log r_f + \log f(\mathrm{Re}) - \tfrac{1}{2}\log\frac{Re^2-\tilde{z}^2}{Re^2} + \frac{1}{2}\right\} - x_f \nabla_f \log r_f - \frac{1}{2}\frac{Re^2-\tilde{z}^2}{Re^2}\frac{\mathrm{Re}^2}{f^2(\mathrm{Re})}\nabla_f \frac{x_f}{r_f}\right]. \tag{21b}$$

The limit $\lim_f$ is defined as the limit $\mathrm{Re}\to 0$ for x_f, y_f and Re fixed. This and similar limit processes will be used very frequently in this paper. In the present case, $f(\mathrm{Re}) = 1$ corresponds to the outer limit and $f(\mathrm{Re}) = \mathrm{Re}$ corresponds to the shank limit. The variables x_f and y_f are intermediate in the sense that

$$\mathrm{Re} < f(\mathrm{Re}) < 1.$$

The matching condition (20) is performed for $\mathrm{Re} < f(\mathrm{Re}) < 1$ in some overlap domain between the shank expansion and the outer expansion. In general, when we apply matching conditions to any two expansions, we first write both of these expansions in terms of variables which are intermediate between the limits of these two expansions such as x_f and y_f used in the present matching. Then we apply the proper matching conditions in some overlap domain and determine the various constants. In the present case, we can see that the matching condition (20) is satisfied if $\epsilon(\mathrm{Re})$ is chosen such that

$$\lim_{\mathrm{Re}\to 0} (-\epsilon \log \mathrm{Re}) = 1, \tag{22a}$$

hence, $\epsilon(\mathrm{Re})$ may be assumed to satisfy a relation

$$-\epsilon \log \mathrm{Re} = 1 + b_1\epsilon + b_2\epsilon^2 + \ldots, \tag{22b}$$

where the b_n can be normalized later. Note that in this case the overlap domain is very small. A sufficiently slow limit is obtained by taking

$$f(\mathrm{Re}) = 1/\log(1/\mathrm{Re}). \tag{22c}$$

Similar matching conditions will be used between $\mathbf{i}$ and $\mathbf{v}_0$ and $\mathbf{i}$ and $\mathbf{w}_0$. We shall omit the details in the subsequent discussion.

Determination of $\mathbf{v}_0$.† In the present case, a solution $\mathbf{l}_1(x_i^+)$ which satisfies the three-dimensional Stokes equations and the boundary condition

$$\mathbf{l}_1(x_i^+) = 0 \quad \text{on} \quad \tau = \tau_0,$$

† The author is indebted to one of the referees for pointing out that $\mathbf{u}_0$, $\mathbf{v}_0$ and $\mathbf{w}_0$ can also be obtained from the Stokes solution for an ellipsoid (cf. Lamb 1932) by certain limit processes.

can be obtained as

$$\mathbf{l}_1(x_i^+) = \tfrac{1}{2}\mathbf{i}[\log(\tau/\tau_0)+1] - \tfrac{1}{2}x^+\nabla^+\log\tau - \tfrac{1}{2}\tau_0\nabla^+(x^+/\tau). \tag{23}$$

Then the intermediate left-end solution is determined as

$$\mathbf{v}_0 = \epsilon(\mathrm{Re})\,\mathbf{l}_1(x_i^+), \tag{24a}$$

where the matching condition

$$\lim_{\substack{f\\ \mathrm{Re}\to 0}} |\mathbf{i}-\mathbf{v}_0| = 0 \tag{24b}$$

for ord f in some overlap domain is also satisfied. Thus $\mathbf{v}_0$ is the correct intermediate solution. The corresponding pressure term is easily determined as

$$p_0' = -\epsilon(\partial\log\tau/\partial x^+). \tag{25}$$

Matching between $\mathbf{u}_0$ *and* $\mathbf{v}_0$. The matching between $\mathbf{u}_0$ and $\mathbf{v}_0$ can be studied by expressing both of them in terms of intermediate variables (i.e. r_β and z_β) which are intermediate between the shank and the left-end variables:

$$r^* = \mathrm{Re}^\beta r_\beta, \quad r^+ = \mathrm{Re}^{\beta-1} r_\beta, \quad \bar{z} = \mathrm{Re}^{2\beta} z_\beta, \quad z^+ = z_\beta \mathrm{Re}^{2(\beta-1)}. \tag{26}$$

They are supposed to match at $z^+\to\infty$ and along $r^{+2}/z^+ = r_\beta^2/z_\beta = r^{*2}/\bar{z} = \text{const.}$ In terms of intermediate variables, the body is a semi-infinite paraboloid and $0<\beta<1$. Thus

$$\tau = \tfrac{1}{4}r_\beta^2/z_\beta + O[\mathrm{Re}^{2(1-\beta)}]. \tag{27}$$

Thus $\mathbf{v}_0 = \epsilon\mathbf{l}_1$

$$= \epsilon\left\{\tfrac{1}{2}\mathbf{i}\left[\log\left(\frac{Re\, r_\beta^2}{2z_\beta}\right)+1\right] - x_\beta\nabla_\beta\log r_\beta - \frac{z_\beta}{Re}\nabla_\beta\left(\frac{x_\beta}{r_\beta^2}\right)\right\}$$

$$-\tfrac{1}{2}\mathbf{k}\,x_\beta\left\{\frac{2}{Re\, r_\beta^2} - \frac{1}{z_\beta}\right\}\epsilon\,\mathrm{Re}^{1-\beta} + O(\epsilon\,\mathrm{Re}^{1-\beta}) \tag{28}$$

and

$$\mathbf{u}_0 = \epsilon\left\{\tfrac{1}{2}\mathbf{i}\left[\log\left(\frac{Re\, r_\beta^2}{2z_\beta}\right)+1\right] - x_\beta\nabla_\beta\log r_\beta - \frac{z_\beta}{Re}\nabla_\beta\left(\frac{x_\beta}{r_\beta^2}\right)\right\}. \tag{29}$$

Therefore

$$\lim_{\substack{f\\ \mathrm{Re}\to 0}} |\mathbf{u}_0-\mathbf{v}_0| = 0 \tag{30}$$

for $f(\mathrm{Re}) = \mathrm{Re}^\beta$ and $0 \leqslant \beta < 1$. In fact they are matched to $O(\mathrm{Re}^{1-\beta}\epsilon)$.

Determination of $\mathbf{w}_0$. Similarly, $\mathbf{w}_0$ is easily determined as

$$\mathbf{w}_0 = \epsilon\{\tfrac{1}{2}\mathbf{i}[\log(\tau^+/\tau_0)+1] - \tfrac{1}{2}x^+\nabla^+\log\tau^+ - \tfrac{1}{2}\tau_0\nabla^+(x^+/\tau^+)\}. \tag{31a}$$

The corresponding pressure term is obtained as

$$p_0'' = -\epsilon\,\partial\log\tau^+/\partial x^+. \tag{31b}$$

The matching between $\mathbf{w}_0$ and $\mathbf{u}_0$ is exactly the same as between $\mathbf{u}_0$ and $\mathbf{v}_0$. The details are not repeated here.

Uniformly valid expansion near the body. Since $\mathbf{u}_0$ matches to $\mathbf{w}_0$ and $\mathbf{v}_0$, a uniformly valid expansion $\mathbf{s}_0$ near the body can, in principle, be obtained. In the present case, we can easily obtain $\mathbf{s}_0$ as follows:

$$\mathbf{s}_0 = \mathbf{u}_0 + \mathbf{v}_0 + \mathbf{w}_0 - \mathbf{v}_0^s - \mathbf{w}_0^s, \tag{32}$$

where $\mathbf{v}_0^s$ is the shank limit of $\mathbf{v}_0$ and $\mathbf{w}_0^s$ is the shank limit of $\mathbf{w}_0$.

Determination of $\mathbf{g}_1$. $\mathbf{g}_1$ must satisfy the governing equation (4) and can be determined by the matching condition that it cancels the unbounded terms of

$$\lim_{f_1}[(\mathbf{u}_0 - \mathbf{i})/\epsilon] = \mathbf{i}\{\log \tilde{r} - \tfrac{1}{2}\log[(Re^2 - \tilde{z}^2)/Re^2] + \tfrac{1}{2} + b_1\} - \tilde{x}\tilde{\nabla}\log\tilde{r}, \tag{33}$$

for $\tilde{r} \to 0$ and $-Re < \tilde{z} < Re$. Similarly, $\mathbf{g}_1$ is supposed to cancel the unbounded term of

$$\lim_{f_2}[(\mathbf{v}_0 - \mathbf{i})/\epsilon] = \tfrac{1}{2}\mathbf{i}\{\log([\bar{z}^2 + \bar{r}^2]^{\frac{1}{2}} - \bar{z}) + 2b_1 + \log 2Re + 1\} - \tfrac{1}{2}\tilde{x}\tilde{\nabla}\log([\bar{r}^2 + \bar{z}^2]^{\frac{1}{2}} - \bar{z}), \tag{34}$$

for $\bar{r}$ and $\bar{z}$ small. Similarly, $\mathbf{g}_1$ has to cancel the unbounded terms of

$$\lim_{f_3}[(\mathbf{w}_0 - \mathbf{i})/\epsilon] \quad \text{for} \quad \tilde{r} \to 0, \quad \text{and} \quad \tilde{z} - Re \to 0.$$

From all these matching conditions, $\mathbf{g}_1$ is determined as follows:

$$\mathbf{g}_1 = -\mathbf{i}\int_{-Re}^{Re} \frac{\exp[-\frac{1}{2}(\{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2\}^{\frac{1}{2}} - \tilde{x})]}{\{(z-\tilde{\zeta})^2 + \tilde{r}^2\}^{\frac{1}{2}}}\,d\tilde{\zeta} + \tilde{\nabla}\int_{-Re}^{Re}\left\{\frac{\exp[-\frac{1}{2}(\{(\tilde{z}-\tilde{\zeta})^2 + \tilde{r}^2\}^{\frac{1}{2}} - \tilde{x})]}{\{(z-\tilde{\zeta})^2 + \tilde{r}^2\}^{\frac{1}{2}}} - \frac{1}{\{(z-\tilde{\zeta})^2 + \tilde{r}^2\}^{\frac{1}{2}}}\right\}d\tilde{\zeta}. \tag{35}$$

For $\tilde{r} \to 0$, $-Re < \tilde{z} < Re$, we have

$$\mathbf{g}_1 = \mathbf{i}(\log\tilde{r} + \gamma - \log 4) - \tilde{x}\tilde{\nabla}\log\tilde{r} + \tfrac{1}{2}\mathbf{i}\{E_1(\tfrac{1}{2}Re - \tfrac{1}{2}\tilde{z}) + E_1(\tfrac{1}{2}Re + \tfrac{1}{2}\tilde{z})\} + \mathbf{k}\left\{\frac{\exp[-\frac{1}{2}(Re-\tilde{z})]}{Re-\tilde{z}} - \frac{1}{Re-\tilde{z}} + \frac{1}{Re+\tilde{z}} - \frac{\exp[-\frac{1}{2}(Re+\tilde{z})]}{Re+\tilde{z}}\right\} + O(\tilde{r}\log\tilde{r}). \tag{36}$$

The function $E_1(x)$ (cf. Erdelyi 1953) is defined as

$$E_1(x) = -Ei(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = -\log\gamma_0 x + e^{-x}\sum_{m=1}^{\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{m}\right)\frac{x^m}{m!}, \tag{37a}$$

where $\gamma = \log\gamma_0 \doteq 0{\cdot}5772$ is Euler's constant. For large value of x, an asymptotic expansion for $E_1(x)$ is

$$E_1(x) = \frac{e^{-x}}{x}\left(1 - \frac{1!}{x} + \frac{2!}{x^2} + \ldots\right). \tag{37b}$$

Thus $E_1(x) \to 0$ as $x \to \infty$ and $E_1(x) \to \log\gamma_0 x$ as $x \to 0$.

Equation (36) therefore shows that $\mathbf{g}_1$ cancels the unbounded terms in equation (33) for $\tilde{r} \to 0$ and $-Re < z < Re$. If we choose all $b_n = 0$ except

$$b_1 = -\tfrac{1}{2} + \gamma - \log 4 \tag{38a}$$

we have the same ϵ as for the two-dimensional case which is

$$\epsilon = 1/\{\log(4/\mathrm{Re}) - \gamma + \tfrac{1}{2}\}. \tag{38b}$$

In addition, for $\tilde{r} \to 0$ and $\tilde{z} \to 0$, we have

$$\mathbf{g}_1 = \tfrac{1}{2}\mathbf{i}\{\log([\tilde{z}^2+\tilde{r}^2]^{\frac{1}{2}} - \tilde{z}) + \gamma - \log 4 + E_1(Re)\} - \tfrac{1}{2}\tilde{x}\tilde{\nabla}\log([\tilde{z}^2+\tilde{r}^2]^{\frac{1}{2}} - \tilde{z}) + \mathbf{k}(e^{-Re}/2Re - 1/2Re) + O(\tilde{z}\log\tilde{z}). \tag{39}$$

Thus it cancels the unbounded term in equation (34). This shows that $\mathbf{g}_1$ is correctly determined as in equation (35).

The corresponding pressure term is

$$p_1^* = -\frac{\partial}{\partial\tilde{x}}\int_{-Re}^{Re}\frac{d\tilde{\zeta}}{\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}}. \tag{40}$$

Determination of $\mathbf{u}_1$. $\mathbf{u}_1$ can be determined by the matching condition

$$\lim_{\substack{f_1 \\ \mathrm{Re}\to 0}} \frac{\mathbf{i}+\epsilon\mathbf{g}_1 - (\mathbf{u}_0+\epsilon\mathbf{u}_1)}{\epsilon} = 0, \tag{41}$$

for ord f_1 in some overlap domain. In the present case, $\mathbf{u}_1$ is then easily determined as

$$\mathbf{u}_1 = \mathbf{u}_1' + \mathbf{u}_1'' \tag{42}$$

with
$$\mathbf{u}_1' = f_1(\tilde{z})\,\mathbf{u}_0, \quad \mathbf{u}_1'' = k_1(\tilde{z})\,\mathbf{u}_0', \tag{43}$$

where
$$\mathbf{u}_0' = \epsilon\mathbf{k}[\log r^* - \tfrac{1}{2}\log\{(Re^2-\tilde{z}^2)/Re^2\}], \tag{44}$$

$$f_1(\tilde{z}) = \tfrac{1}{2}\,[E_1(\tfrac{1}{2}Re - \tfrac{1}{2}\tilde{z}) + E_1(\tfrac{1}{2}Re + \tfrac{1}{2}\tilde{z}) + \log(Re-\tilde{z}) + \log(Re+\tilde{z}) - 2\log Re] \tag{45a}$$

and
$$k_1(\tilde{z}) = \frac{\exp[-\frac{1}{2}(Re-\tilde{z})]}{Re-\tilde{z}} - \frac{1}{Re-\tilde{z}} + \frac{1}{Re+\tilde{z}} - \frac{\exp[-\frac{1}{2}(Re+\tilde{z})]}{Re+\tilde{z}}. \tag{45b}$$

It can easily be shown that $\mathbf{u}_0'$ satisfies the governing equation (8c) and that $\mathbf{u}_1$ satisfies the governing equations (8a–d) and $\mathbf{u}_1 = 0$ on the body. By expressing $\mathbf{u}_1$ in terms of outer variables, the matching condition is obviously satisfied and therefore $\mathbf{u}_1$ is correctly determined. Thus we have

$$p_1 = -\epsilon(2f_1(\tilde{z})\,x^*/r^{*2}). \tag{46}$$

Determination of $\mathbf{v}_1$ *and* $\mathbf{w}_1$. $\mathbf{v}_1$ can be determined by the matching condition that

$$\lim_{\substack{f_2 \\ \mathrm{Re}\to 0}} \frac{\mathbf{i}+\epsilon\mathbf{g}_1 - (\mathbf{v}_0+\epsilon\mathbf{v}_1)}{\epsilon^2} = 0, \tag{47}$$

for ord f_2 in some overlap domain. In the present case, $\mathbf{v}_1$ can be obtained as follows:

$$\mathbf{v}_1 = \mathbf{v}_1' + \mathbf{v}_1'', \tag{48}$$

with
$$\mathbf{v}_1' = c_1\mathbf{v}_0 = \tfrac{1}{2}\{E_1(Re) - \log Re - \gamma + \log 4\}\,\mathbf{v}_0, \tag{49a}$$

$$\begin{aligned}\mathbf{v}_1'' = D_1\mathbf{v}_1' &= (e^{-Re}/2Re - 1/2Re)\,\mathbf{v}_0' \\ &= (e^{-Re}/2Re - 1/2Re)\,\tfrac{1}{2}\epsilon\{\mathbf{k}\log(\tau/\tau_0) - \nabla^+(\tau - \tau_0\log\tau)\}.\end{aligned} \tag{49b}$$

It can easily be shown that $\mathbf{v}_1$ satisfies the three-dimensional Stokes equations and the matching condition (47).

The corresponding pressure term is obtained as

$$p_1' = -\tfrac{1}{2}\epsilon[E_1(Re) - \log Re - \gamma + \log 4]\,\partial \log \tau/\partial x^+ + \tfrac{1}{2}\epsilon[1/2Re - e^{-Re}/2Re]\,\partial \log \tau/\partial z^+. \tag{50}$$

Similarly, by symmetry and by the matching condition that

$$\lim_{\substack{f_2 \\ \mathrm{Re}\to 0}} \frac{\mathbf{i} + \epsilon \mathbf{g}_1 - (\mathbf{w}_0 + \epsilon \mathbf{w}_1)}{\epsilon^2} = 0, \tag{51}$$

for $\operatorname{ord} f_2$ in some overlap domain, $\mathbf{w}_1$ and p_1'' can be easily determined. The details are not repeated here.

Matching between $\mathbf{u}_1$ *and* $\mathbf{v}_1$ (*or* $\mathbf{w}_1$). If we write $\mathbf{u}_1$ and $\mathbf{v}_1$ (or $\mathbf{w}_1$) in terms of intermediate variables, we can easily show that they are matched to $O(\mathrm{Re}^{\alpha}\epsilon)$ and $0 < \alpha < 1$. The details are omitted. Since $\mathbf{u}_1$ matches to $\mathbf{v}_1$ (or $\mathbf{w}_1$), $\mathbf{g}_2$ can be determined by matching with them.

Determination of $\mathbf{g}_2$. It can be seen from equation (5*a*, *b*) that the governing equation for $\mathbf{g}_2$ is a non-homogeneous Oseen equation. In general, $\mathbf{g}_2$ can be divided into three parts to be determined separately.

$$\mathbf{g}_2 = \mathbf{g}_2' + \mathbf{g}_2'' + \mathbf{g}_2'''; \tag{52}$$

$\mathbf{g}_2'''$ is the particular solution of Oseen equations and is

$$\mathbf{g}_2''' = \iiint_{-\infty}^{\infty} t_{ij}(\tilde{x}_k - \tilde{\xi}_k)\, f_j(\tilde{\xi}_k)\, d\tilde{\xi}_1\, d\tilde{\xi}_2\, d\tilde{\xi}_3, \tag{53}$$

where

$$\mathbf{f}(\tilde{x}_i) = (\mathbf{g}_1 . \tilde{\nabla})\,\mathbf{g}_1 = -\mathbf{g}_1 \times \operatorname{curl} \mathbf{g}_1 + \tfrac{1}{2}\tilde{\nabla} \mathbf{g}_1^2 \tag{54}$$

and t_{ij} is the fundamental solution of Oseen equations,

$$t_{ij}(\tilde{x}_k) = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} = \begin{pmatrix} \partial A/\partial \tilde{x} & \partial A/\partial \tilde{y} & \partial A/\partial \tilde{z} \\ \partial A/\partial \tilde{y} & 0 & 0 \\ \partial A/\partial \tilde{z} & 0 & -\partial A/\partial \tilde{x} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial B/\partial \tilde{y} & \partial B/\partial \tilde{z} \\ 0 & \partial B/\partial \tilde{z} & -\partial B/\partial \tilde{y} \end{pmatrix} + \frac{\exp[-\frac{1}{2}(\tilde{R}-\tilde{x})]}{4\pi \tilde{R}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{55}$$

Here

$$\left.\begin{aligned} A &= \frac{1}{4\pi}\left[\frac{1}{\tilde{R}} - \frac{\exp[-\frac{1}{2}(\tilde{R}-\tilde{x})]}{\tilde{R}}\right], \\ B &= -(1/4\pi)[1 - \exp\{-\tfrac{1}{2}(\tilde{R}-\tilde{x})\}]\,\partial \log(\tilde{R}-\tilde{x})/\partial \tilde{y}, \\ \tilde{R}^2 &= \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2. \end{aligned}\right\} \tag{56}$$

Then for the same reason as discussed by Kaplun & Lagerstrom (1957), $\mathbf{g}_2'''$ has no contribution to drag calculation to the order of ϵ^2.

$\mathbf{g}_2'$ is the solution of the homogeneous Oseen solution and is determined by the matching condition that it cancels the unbounded term of

$$\lim_{\substack{f_1 \\ \mathrm{Re}\to 0}} \frac{\mathbf{i} + \epsilon \mathbf{g}_1 - (\mathbf{u}_0 + \epsilon \mathbf{u}_1')}{\epsilon^2} = f_1(\tilde{z})\left\{\mathbf{i}\left(\log \tilde{r} - \tfrac{1}{2}\log \frac{Re^2 - \tilde{z}^2}{Re^2} + \tfrac{1}{2} + b_1\right) - \tilde{x}\tilde{\nabla} \log \tilde{r}\right\}, \tag{57}$$

for $\tilde{r} \to 0$ and $-Re < \tilde{z} < Re$ and ord f_1 in some overlap domain. For $\bar{z} \to 0$ and $\tilde{r} \to 0$, $\mathbf{g}_2'$ has to cancel the unbounded terms of

$$\lim_{\substack{f_2 \\ \mathrm{Re}\to 0}} \frac{\mathbf{i}+\epsilon\mathbf{g}_1-(\mathbf{v}_0+\epsilon\mathbf{v}_1')}{\epsilon^2} = c_1\{\tfrac{1}{2}\mathbf{i}[\log(\{\bar{z}^2+\tilde{r}^2\}^{\frac{1}{2}}-\bar{z}) + 2b_1+\log 2Re+1] - \tfrac{1}{2}\tilde{x}\tilde{\nabla}\log(\{\bar{z}^2+\tilde{r}^2\}^{\frac{1}{2}}-\bar{z})\}. \quad (58)$$

Similarily, for $\tilde{r} \to 0$ and $\bar{z} \to 0$, it cancels the unbounded terms of

$$\lim\{[\mathbf{i}+\epsilon\mathbf{g}_1-(\mathbf{w}_0+\epsilon\mathbf{w}_1')]/\epsilon^2\}.$$

By these matching conditions, $\mathbf{g}_2'$ is determined as follows:

$$\mathbf{g}_2' = -\tfrac{1}{2}\mathbf{i}\int_{-Re}^{Re}\left[E_1\left(\frac{Re+\tilde{\zeta}}{2}\right)+E_1\left(\frac{Re-\tilde{\zeta}}{2}\right)+\log(Re-\tilde{\zeta}) + \log(Re+\tilde{\zeta})-2\log Re\right]\frac{\exp[-\frac{1}{2}(\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}-\tilde{x})]}{\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}}d\tilde{\zeta}$$
$$+\tfrac{1}{2}\tilde{\nabla}\int_{-Re}^{Re}\left[E_1\left(\frac{Re+\tilde{\zeta}}{2}\right)+E_1\left(\frac{Re-\tilde{\zeta}}{2}\right)+\log(Re+\tilde{\zeta})+\log(Re-\tilde{\zeta}) -2\log Re\right]\left[\frac{\exp[-\frac{1}{2}(\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}-\tilde{x})]}{\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}}-\frac{1}{\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}}\right]d\tilde{\zeta}. \quad (59)$$

It can be shown that $\mathbf{g}_2'$ is correctly determined. The details are given in Shi (1963). Similarly $\mathbf{g}_2''$ can be determined by the matching with $\mathbf{u}_1''$, $\mathbf{v}_1''$ and $\mathbf{w}_1''$, respectively. Since $\mathbf{u}_1''$, $\mathbf{v}_1''$, and $\mathbf{w}_1''$ are symmetrical with respect to the axis of revolution of the ellipsoid, $\mathbf{g}_2''$ will make no contribution to the drag calculation. Thus details of the determination of $\mathbf{g}_2''$ will not be presented here.

A higher-order solution can be obtained by the use of a matching procedure similar to that discussed above.

Uniformly valid expansion. From the intermediate expansions and the outer expansions one can construct an expansion which is uniformly valid for the entire flow field. Now the first term $\mathbf{q}_0(\tilde{x}_i, \mathrm{Re})$, uniformly valid to order unity can be constructed by considering $\mathbf{v}_0$, $\mathbf{u}_0$, $\mathbf{w}_0$, $\mathbf{g}_0$ and $\mathbf{g}_1$. From (19), (24), (31) and (35), $\mathbf{q}_0(\tilde{x}_i, \mathrm{Re})$ can be constructed into the following simple form:

$$\mathbf{q}_0(\tilde{x}_i, \mathrm{Re}) = \mathbf{i}-\epsilon\mathbf{i}\int_{-c}^{c}\frac{\exp[-\frac{1}{2}(\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}-\tilde{x})]}{\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}}d\tilde{\zeta} + \epsilon\tilde{\nabla}\int_{-c}^{c}\left[\frac{\exp[-\frac{1}{2}(\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}-\tilde{x})]}{\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}}-\frac{1}{\{(\tilde{z}-\tilde{\zeta})^2+\tilde{r}^2\}^{\frac{1}{2}}}\right]d\tilde{\zeta}, \quad (60)$$

where $c = Re-\mathrm{Re}^2/2Re$. It should be noted that in the present case the source distribution for $\mathbf{q}_0$ is constant and uniformly distributed within

$$-Re+\mathrm{Re}^2/2Re \leqslant \tilde{z} \leqslant Re-\mathrm{Re}^2/2Re.$$

The sources are inside the ellipsoid cylinder. But in the outer limit, the source is uniformly distributed within $-Re \leqslant \tilde{z} \leqslant Re$ and the source comes to the surface of the ellipsoid at the two ends. This shows that the outer limit cannot be valid near the two ends and the two end limits must be introduced.

Computation of drag force. The drag force on the ellipsoid can be obtained either by calculating the viscous stresses on the ellipsoid or by the momentum integral. In the present case, it can be obtained by comparison with the fundamental solutions of the Oseen equations (cf. Lagerstrom 1956). The total drag force is found to be of the order of ϵ where

$$\epsilon = 1/\{\log(4/\mathrm{Re}) - \gamma + \tfrac{1}{2}\}. \tag{61}$$

The drag force can be obtained to the order of ϵ either by calculating the viscous stress on the ellipsoid using $\mathbf{u}_0$, $\mathbf{v}_0$, and $\mathbf{w}_0$ and the corresponding pressures or by comparing $\mathbf{g}_1$ with the fundamental solutions of the Oseen equations. By comparing $\mathbf{g}_1$ with the fundamental solutions of the Oseen equations, we can see that to the order of ϵ, the drag force is constant along the ellipsoid. The drag per unit length is the same as that of a two-dimensional circular cylinder; thus the total drag

$$D_1 = 8\pi\mu UL\epsilon. \tag{62}$$

The drag forces can be obtained to the order of ϵ^2 by calculating the viscous stresses and the pressure on the ellipsoid using $\mathbf{v}_1$, $\mathbf{u}_1$ and $\mathbf{w}_1$. But since $\mathbf{u}_1''$, $\mathbf{v}_1''$ and $\mathbf{w}_1''$ are symmetrical with respect to the axis of revolution of the ellipsoid, they make no contribution to the drag calculation. Thus the drag can be obtained by considering $\mathbf{u}_1'$, $\mathbf{v}_1'$ and $\mathbf{w}_1'$ only. Since $\mathbf{u}_1'$, $\mathbf{v}_1'$ and $\mathbf{w}_1'$ are matched to $\mathbf{g}_2'$, the drag force can also be obtained by comparing $\mathbf{g}_2'$ with the fundamental solutions of the Oseen equations. The results are the same for both calculations. In the present case, we can see from $\mathbf{g}_2'$ that the drag force is no longer constant along the cylinder to the order of ϵ^2. The variation of the singular drag force is easily found as

$$f_1(\tilde{z}) = \tfrac{1}{2}[E_1(\tfrac{1}{2}Re + \tfrac{1}{2}\tilde{z}) + E_1(\tfrac{1}{2}Re - \tfrac{1}{2}\tilde{z}) + \log(Re + \tilde{z}) + \log(Re - \tilde{z}) - 2\log Re]. \tag{63}$$

The last three terms in the above equation depend upon the shape of the body. Thus the existence of ends and the variation of the diameter along the axis of revolution effects the drag to the order of ϵ^2. The total drag force of order ϵ^2 can be obtained by integration and is

$$D_2 = 8\pi\mu UL\{E_1(Re) + (\log 2 - 1) + (1 - e^{-Re})/Re\}. \tag{64}$$

Then the total drag is

$$D = 8\pi\mu UL\{\epsilon + \epsilon^2[E_1(Re) + (\log 2 - 1) + (1 - e^{-Re})/Re] + O(\epsilon^3)\}, \tag{65}$$

or

$$C_D = (4\pi/\mathrm{Re})\{\epsilon + \epsilon^2[E_1(Re) + \log 2 - 1 + (1 - e^{-Re})/Re] + O(\epsilon^3)\}. \tag{66}$$

The corresponding C_D for a two-dimensional cylinder (cf. Kaplun 1957) is

$$C_D = (4\pi/\mathrm{Re})\{\epsilon - 0{\cdot}87\epsilon^3 + O(\epsilon^4)\}. \tag{67}$$

5. Results and discussion

In this paper, solutions for viscous flow past an ellipsoid of revolution of large aspect ratio are obtained. The results are valid for small Re and fixed *Re*. The expansion procedures established here should also be valid for other kinds of cylinders of large aspect ratio. Although the body is three-dimensional the expansions are in inverse powers of logarithms of the Reynolds number as in the case of a two-dimensional cylinder. The drag formula, obtained to the order of ϵ^2, shows that the leading term is the same as that for a two-dimensional circular cylinder. Thus the variation of the diameter along the axis of revolution of the ellipsoid has no effect to this order. The difference to the order of ϵ^2 is due to the variation of diameter along the cylinder. This can be seen from the matching between the shank expansion and the outer expansion. If the body is $r^* = f(\tilde{z})$ instead of the ellipsoid but has the same type of ends as the ellipsoid, Shi (1963) has shown that the singular drag force variation of order ϵ^2 is

$$\tfrac{1}{2}[E_1(\tfrac{1}{2}Re+\tfrac{1}{2}\tilde{z})+E_1(\tfrac{1}{2}Re-\tfrac{1}{2}\tilde{z})+2\log f(\tilde{z})]. \tag{68}$$

From the above formula, it follows that the cylinder which makes the drag of order ϵ^2 equal to zero is

$$r^* = f(\tilde{z}) = \exp[-\tfrac{1}{2}E_1(\tfrac{1}{2}Re+\tfrac{1}{2}\tilde{z})-\tfrac{1}{2}E_1(\tfrac{1}{2}Re-\tfrac{1}{2}\tilde{z})]. \tag{69}$$

For $\tilde{z} \to -Re$ or $\bar{z} = Re+\tilde{z} \to 0$, we have

$$r^{+2} = C^2 z^+ + O(\mathrm{Re}^2), \tag{70}$$

where

$$C = (\tfrac{1}{2}\gamma_0)^{\frac{1}{2}}\exp[-\tfrac{1}{2}E_1(Re)]. \tag{71}$$

This cylinder also has paraboloidal ends. By actual numerical plotting for large Re, this cylinder has a constant radius region in the centre portion of the cylinder. In fact, for Re approaching infinity, the cylinder is of almost constant radius except near the ends. The drag of this cylinder agrees with the two-dimensional case to order ϵ^2. This example should help to verify that the variation of diameter along the axis of revolution has an effect of order ϵ^2 on the drag force. Finally, the solutions obtained in this paper are valid for large aspect ratio and fixed Re. Naturally there arise some questions about the relation between the solutions obtained in this paper and those obtained for small Re and finite aspect ratio. These questions may best be answered from the drag standpoint. Without going into details, one can obtain the following drag formula from Lamb (1932) for Stokes flow past an ellipsoid of revolution with its axis of revolution perpendicular to the uniform flow at infinity:

$$C_D = \frac{8\pi}{\mathrm{Re}}\left[\frac{A}{A^2-1}+\frac{2A^2-3}{2(A^2-1)^{\frac{3}{2}}}\log\left\{\frac{A+(A^2-1)^{\frac{1}{2}}}{A-(A^2-1)^{\frac{1}{2}}}\right\}\right]^{-1}. \tag{72}$$

Here $A = L/\lambda$ is the aspect ratio. The above formula was also obtained by Cole & Roshko (1954). For large aspect ratios, the above formula reduced to the following formula

$$C_D = 4\pi/\mathrm{Re}\,[\log(2L/\lambda)+\tfrac{1}{2}], \tag{73}$$

when $L/\lambda \gg 1$ but $Re \ll 1$. The above formula was first obtained by Oberbeck (1876).† This formula may be written

$$\mathrm{Re}\,C_D = 4\pi/[\{\log(4/\mathrm{Re})-\gamma+\tfrac{1}{2}\}+\{\log Re+\gamma-\log 2\}], \tag{74}$$

which is the same as equation (66) when the $\log Re/\log(4/\mathrm{Re})$ is small. From equation (72) one can study the effects of aspect ratio from order unity to large

† The author is indebted to one of the referees for pointing out this formula and suggesting that this paper conclude with a discussion of this aspect of the problem.

value as long as both Re and Re are small. But physically there will be cases where Re is not small. The two-dimensional case corresponds to having L or Re approach infinity. Intuitively one can feel **that in** order to connect the two- and three-dimensional solutions, the case of finite Re must be studied. The reason is that in the two-dimensional case, the cylinder is infinite in both the outer limit and the inner limit. Now if Re is small, the ellipsoid shrinks to a point at the outer limit and then no matter what the aspect ratio is, the point can never produce the disturbance to the outer flow that an infinite cylinder does. Since the Oberbeck formula was obtained from solutions of Stokes's equations for large aspect ratios, theoretically it is only valid when both Re and Re are small. Whether the Oberbeck formula is valid for large Re is subject to question. In fact, for large L or Re the Oberbeck formula encounters trouble both mathematically and physically. In order to compare it with the two-dimensional solution for a cylinder of the same radius in the same viscous fluid, we assume Re is small and both Re and λ are fixed. Then if the length of the cylinder is increased to infinity the Oberbeck formula predicts that the drag is zero. Obviously this is not physically possible. On the other hand it is not mathematically consistent in that it does not approach the two-dimensional drag formula obtained by Proudman & Pearson (1957) and Kaplun & Lagerstrom (1957) as the length of the cylinder approaches infinity. Therefore, there exists a certain region in which the Oberbeck formula is not valid. The results obtained in the present paper seem to fill this gap. For Re small, the drag formula obtained in equation (66) becomes

$$\mathrm{Re}\, C_D = 4\pi\{\epsilon + \epsilon^2(-\log Re - \gamma + \log 2) + O(\epsilon^3)\}. \tag{75}$$

This matches the Oberbeck formula to order ϵ^3 in the region where both Re and $\log Re/\log(4/\mathrm{Re})$ are small. There exists an overlap domain between these two formulas. Furthermore, the drag obtained from our formula is not zero as L and Re approach infinity. In fact, the leading term is the drag obtained by Proudman & Pearson (1957) and Kaplun & Lagerstrom (1957) for a two-dimensional circular cylinder. By comparing our formula with the Oberbeck formula, we can see that those terms which are small when Re is small are no longer negligible when Re is finite. These terms result from interaction with the outer flow (i.e. $\mathbf{g}_1$). Therefore the Stokes solution is not valid for Re finite because it does not take into consideration the outer expansion which is important when Re is no longer small. Now by combining (66) and (72) we are able to study the effect of aspect ratio for any aspect ratio, for any length of the ellipsoid and any value of Re as long as Re is small. At least from the drag standpoint, the results obtained in the present paper clarify the relation of drag force obtained from the three-dimensional limit which is valid for finite aspect ratio and small Re and the drag force obtained from the two-dimensional limit which is valid for infinite aspect ratio and infinite Re. In principle, the effect of aspect ratio from order of unity to infinity can be drawn by using (66) and (72).

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